Institute of Actuaries of India

ACET December 2018 Solutions

Mathematics

1. B. The given matrix is $\begin{pmatrix} 1 & 5 & 7 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. The characteristic equation is $|A - \lambda I| = 0$, giving $\begin{vmatrix} 1-\lambda & 5 & 7 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$, that is, $(1-\lambda)(2-\lambda)(3-\lambda) = 0$. Hence the characteristic roots are 1, 2, 3.

2. A. The given matrix $\begin{pmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{pmatrix}$ is singular. Hence $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = 0$, that is, $(1-x^3) - x(x^2 - x^2) + x^2(x^4 - x) = 0$ giving $1 - 2x^3 + x^6 = (1-x^3)^2 = 0$, that is, $1 - x^3 = 0$ or $x^3 = 1$. The possible real value of x is 1.

3. C. $\log_9(x+1) = \log_3(x-1)$, $x > 1$ implies $\frac{\log(x+1)}{\log 9} = \frac{\log(x-1)}{\log 3}$, that is, $\log(x+1) = \log(x-1)^2$, implying $(x+1) = (x-1)^2$ giving $x^2 - 3x = 0$. Hence $x = 3$ or $x = 0$. Since $x \neq 0$, we have $x = 3$.

4. B. $\frac{2x+3}{(x+2)^2(x+1)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$ implies

$$2x + 3 = A(x+2)^2 + B(x+1)(x+2) + C(x+1).$$

   This is an identity that should hold for all x. Putting $x = -1$ we have $A = 1$.

5. A. Put $x = \sqrt{2\sqrt{2\sqrt{2\ldots}}}$. Hence, $x = \sqrt{2x}$ or $x^2 = 2x$, giving $x = 0$ or $x = 2$. Let $x_1 = \sqrt{2}$, $x_n = \sqrt{2x_{n-1}}$.

   The sequence x_n is increasing and positive (this can be proved by induction). Therefore, its limit x cannot be 0. Therefore, $x = 2$.

   Alternatively, let $y_n = \log x_n$. Then $y_1 = \frac{1}{2}\log 2$, $y_n = \frac{1}{2}\log 2 + \frac{1}{2}y_{n-1}$. Therefore, $(y_n - \log 2) = \frac{1}{2}(y_{n-1} - \log 2)$, i.e., $(y_n - \log 2) \to 0$, $y_n \to \log 2$ and $x_n \to 2$.

6. C. $\sin 105^\circ = \sin(60^\circ + 45^\circ) = \sin 60^\circ \cos 45^\circ + \sin 45^\circ \cos 60^\circ = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$.

7. B. The general term is $k(k+1)(k+2) = k^3 + 3k^2 + 2k$. Thus, the summation of the series is

$$\sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k = \left(\frac{n(n+1)}{2}\right)^2 + 3\frac{n(n+1)(2n+1)}{6} + n(n+1)$$

$$= n(n+1)\left[\frac{n(n+1)}{4} + \frac{2n+1}{2} + 1\right]$$

$$= n(n+1)\left[\frac{n(n+1)+4n+2+4}{4}\right]$$

$$= n(n+1)\left[\frac{n^2+5n+6}{4}\right]$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}.$$

8. B. In the quadratic equation the discriminant $b^2 - 4ac = 1 - 4 \times 2 \times 1 = -7$. Hence the roots are complex conjugates (hence unequal).

9. A. $\frac{x^2-2x-1}{x+1} < x$ implies $\frac{x^2-2x-1-x(x+1)}{x+1} < 0$, that is, $\frac{-3x-1}{x+1} < 0$. Equivalently, it is sufficient to consider the inequality $-(3x+1)(x+1) < 0$. This means EITHER $(3x+1) > 0$ and $(x+1) > 0$ OR $(3x+1) < 0$ and $(x+1) < 0$. Hence, $x > -\frac{1}{3}$ or $x < -1$.

Alternative method. Case I: $x + 1 > 0$. In this case, $x^2 - 2x - 1 < x(x+1)$, that is, $x > -1/3$. If this condition holds, the condition $x + 1 > 0$ holds automatically.

Case II: $x + 1 < 0$. In this case, $x^2 - 2x - 1 > x(x+1)$, that is, $x < -1/3$. This condition is superfluous because $x < -1$ in this case.

By combining the two cases, we have $(x > -1/3) \cup (x < -1)$.

10. D. The given vector is $3\vec{\imath} + 4\vec{\jmath} - 12\vec{k}$, which has magnitude $\left|3\vec{\imath} + 4\vec{\jmath} - 12\vec{k}\right| = \sqrt{3^2 + 4^2 + (-12)^2} = 13$. Hence, the unit vector in the direction of $3\vec{\imath} + 4\vec{\jmath} - 12\vec{k}$ is $\frac{1}{13}(3\vec{\imath} + 4\vec{\jmath} - 12\vec{k})$.

11. A. By L'Hospital rule, the given limit is equal to $n(3^{n-1})$, which has to coincide with 108. Since $n(3^{n-1})$ is an increasing function of n, there can be at most one integer solution to the equation $n(3^{n-1}) = 108$. As $3^4 > 108/5$, the solution, if any, is less than 5. By trial and error we have $n = 4$.

12. D. $u = \log\left(\frac{x^2+y^2}{xy}\right)$. $\frac{\partial u}{\partial x} = \frac{xy}{x^2+y^2} \cdot \frac{xy\,(2x)-(x^2+y^2)y}{(xy)^2} = \frac{x^2-y^2}{x(x^2+y^2)}$. Similarly $\frac{\partial u}{\partial y} = \frac{y^2-x^2}{y(x^2+y^2)}$. Hence, $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \ = x\frac{x^2-y^2}{x(x^2+y^2)} + y\frac{y^2-x^2}{y(x^2+y^2)} = 0$.

13. A. Let $(x) = \frac{\sin x}{2+\cos x}$. Then $f(-x) = \frac{\sin(-x)}{2+\cos(-x)} = \frac{-\sin x}{2+\cos x} = -f(x)$. Hence, $f(x)$ is an odd function. Therefore, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin x}{2+\cos x}dx = 0$.

Alternatively, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin x}{2+\cos x}dx = -\log(2+\cos x)|_{-\pi/2}^{\pi/2} = 0$.

14. B. Put $\sqrt{x} = t$, which implies $t^2 = x$ and $2tdt = dx$. Hence $\int \frac{1}{x+\sqrt{x}}\,dx = \int \frac{2t}{t^2+t}\,dt = 2\log(1+t) + c = 2\log\left(1+\sqrt{x}\right) + c$.

15. B. $\int_{-1}^{1}(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots)dx = \int_{-1}^{1} e^x dx = e^1 - e^{-1}$.

16. D. $\int_0^{\frac{\pi}{2}} \sin^7 x\,dx = \int_0^{\frac{\pi}{2}}(1-\cos^2 x)^3 \sin x\,dx = \int_0^1 (1-y^2)^3 dy = \int_0^1 (1 - 3y^2 + 3y^4 - y^6)dy = 1 - 1^3 + 3\frac{1^5}{5} - \frac{1^7}{7} = \frac{21-5}{35} = \frac{16}{35}$.

Alternatively, $\int_0^{\frac{\pi}{2}} \sin^n x\ dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}$ when n is odd. Hence, $\int_0^{\frac{\pi}{2}} \sin^7 x\ dx = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{16}{35}$.

17. C. Let us use the parabolic function $f(x) = ax^2 + bx + c$. Solving the three given equations, we get $a = -2, b = 11, c = -3$. Hence, $f(6) = -9$.

Statistics

18. C. From the Box-Whisker plot, it is seen that $Q_1 = 75,\ Q_2 = 95\ , Q_3 = 125.$
The Inter-Quartile Range, $Q_3 - Q_1 = 125 - 75 = 50.$
Median, $Q_2 = 95.$

19. B. The sum of probabilities of all the positive integers should be 1, that is,

$$ce^{-3}\sum_{n=1}^{\infty}\frac{3^n}{n!} = ce^{-3}\left(\sum_{n=0}^{\infty}\frac{3^n}{n!} - 1\right) = ce^{-3}(e^3 - 1) = 1.,$$

Therefore, $c = 1/(1 - e^{-3})$.

20. D. $\frac{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}}{n+1} = \frac{(1+1)^n}{n+1} = \frac{2^n}{n+1}.$

21. D. Since there are $\binom{6}{2}$ ways of choosing two men, and for each of these choices there are $\binom{12}{3}$ ways of choosing three women, the answer is $\frac{\binom{6}{2}\binom{12}{3}}{\binom{18}{5}}$.

22. A. Let A be the event that a student registers for Elective A,
B be the event that a student registers for Elective B,
M be the event that a student happens to be male,
F be the event that a student happens to be female.
Then $P(M) = 0.7$ and $P(F) = 0.3$.
Also $A = A \cap (M \cup F)$, implying

$$\begin{aligned} P(A) &= P(A \cap M) + P(A \cap F) = P(A|M)P(M) + P(A|F)P(F) \\ &= (0.7 \times 0.8) + (0.3 \times 0.6) = 0.74. \end{aligned}$$

Similarly, $P(B) = 0.26$.
The ratio of students registering for courses A and B is 74: 26.

23. B. By putting the integral of the pdf equal to 1, we have $c = 1/4$. Identifying the pdf as that of the uniform distribution, the variance is $4/3$.

Alternatively the variance can be computed from the definition.

24. C. By definition,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+P(A \cap B \cap C).$$

Since $P(B \cap C) = 0$, we have $P(A \cap B \cap C) = 0$.
$P(A \cup B \cup C) = \frac{1}{4} + \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} - 0 + 0 = \frac{13}{24}$.

25. B. Follows from the definition of independence.

26. B. $E(Y) = aE(X) - b = 0$ implies $10a - b = 0$.
$V(Y) = a^2V(X) = 1$ implies $25a^2 = 1$, giving $a = \frac{1}{5}$ (since $a > 0$), $b = 2$.

27. C. The pdf $(x) = \frac{1}{2}\, e^{-|x|}$, $-\infty < x < \infty$. Therefore,

$$f(x) = \begin{cases} \frac{1}{2}e^{x}, & x < 0, \\ \frac{1}{2}e^{-x}, & x \geq 0. \end{cases}$$

$$F(x) = P(X \leq x) = \begin{cases} \int_{-\infty}^{x} \frac{1}{2}e^{t}dt, & x < 0, \\ \frac{1}{2} + \int_{0}^{x} \frac{1}{2}e^{-t}dt, & x \geq 0. \end{cases}$$

$$= \begin{cases} \frac{e^{x}}{2} & \text{if } x < 0, \\ \frac{1}{2} + \frac{1}{2}(1 - e^{-x}) & \text{if } x \geq 0. \end{cases}$$

28. D. $m\left[E\left(\frac{X-\mu}{\sigma}\right)\right]^2 + n\,V\left(\frac{X-\mu}{\sigma}\right) = m \cdot 0^2 + n \cdot 1 = n.$

29. C. For the Binomial distribution, Mean $np = 8$, Variance $npq = 6$. This gives $=$ $3/4\,, p = 1/4$. Thus, we have $np = n/4 = 8$, implying $n = 32$.

30. A. Mean of the Exponential random variable is 2. Hence variance is 4.
Coefficient of Variation $= \frac{\text{standard deviation}}{\text{mean}} = 1.$

31. A. The regression coefficient is $b_{yx} = \frac{rs_y}{s_x} = 0.8 \times \frac{3.5}{2.5} = 1.12$.

Data Interpretation

32. D. $P = \{a, b, d, f\}$, $Q = \{b, d, c, e\}$, $R = \{d, e, f, g\}$. Therefore, $Q \cap R = \{d, e\}$, $Q \cap P = \{b, d\}$, $P \cap R = \{d, f\}$ and $(P \cap R) \cup (P \cap Q) \cup (Q \cap R) = \{b, d, e, f\}$.

33. B. $D = C^2 - (A + B)^2 = (A + 2B)^2 - (A + B)^2 = B(2A + 3B) = (A + 1)(5A + 3) = 5A^2 + 8A + 3$. Therefore, $D = 304$ means $5A^2 + 8A - 301 = 0$, i.e., $(A - 7)(5A + 43) = 0$. Since A has to be a positive integer, the only feasible solution is $A = 7$.

34. A. Maximum production capacity = Production/ Capacity utilization.

    AA: $(3.5/0.77) = 4.55$,
    BB: $(2.8/0.73) = 3.84$,
    CC: $(1.6/0.66) = 2.42$,
    DD: $(1.5/0.61) = 2.46$.

35. B. Unutilised capacity = Production capacity * (1- Capacity utilization) = (Production/ Capacity utilization) * (1- Capacity utilization).

    $= (1.6/0.66) \times 0.34 \times 1000 = 2.42 \times 0.34 \times 1000 = 824$ tonnes.

36. B. Average selling price = Total sales value / total sales = Rs. 138 Crores per 10,000 tonnes = Rs. 1.38 lakh per tonne.

37. C. The absolute difference is 70 for April compared to 65 for May over their previous months.

38. D. 3 increases and 2 decreases.

English

39. C.

40. D.

41. A.

42. D.

43. C.

44. B.

45. B.

46. D.

47. A.

48. C.

49. D.

50. B.

51. D.

52. C.

53. B.

54. D.

55. D.

56. D.

57. D.

58. B.

59. C.

60. D.

61. B.

62. A.

Logical Reasoning

63. D. Son of Y's son -- Grandson; Brother of Y's grandson -- Y's grandson.

64. D. We can't determine the last day of the next month which has 31 days because we don't know the days in present month which may be 30 or 31. And the next month may be March containing 31 days, which implies that the present month may be February which contains 28 or 29 days.

65. B. Sequence of body parts from top to bottom.

66. A. At 3 O'clock, Minute hand is at 12 while the Hour hand is at 3. The angle between them is 90°. The minute hand has to sweep through $30 \times 6 = 180°$ for reaching the figure 6 to show 30 minutes. Simultaneously the Hour hand, which moves $30°$ every 60 minutes, will also rotate for 30 minutes. Thus, starting from the mark 3, the hour hand will cover an angle $30 \times 30/60 = 15°$. Hence, the angle between the Hour and the Minute hands is $180 - 90 - 15 = 75°$.

67. B. The cube has eight corners. Only the corner sub-cubes would have three faces painted blue.

68. B. Nothing about the growth of economy is mentioned in the statement. So, I does not follow. Also, it is given that forty per cent of national income is shared by five per cent of households. This indicates unequal distribution. So, II follows.

69. B. Let's suppose Amar is a truth teller. Then according to Amar, Akbar is a liar. Hence Anthony would be alternator i.e. one of his statements should be true and others should be false. But in this case, both of his statements are false. Hence Amar is not the truth teller.

    If Akbar is the truth teller, then Anthony should be a liar, but his statement is true.

    If Anthony is truth teller, then according to him, Amar is a liar and Akbar is the Electrician and an alternator. And we can verify that Akbar's first statement is true and second is false. Both the statements made by Amar are indeed false. Hence this assumption is true and Akbar is the Electrician.

70. C. If one adds up the numbers of people speaking the three different languages $(6 + 15 + 6)$, two persons are double-counted and one person is triple-counted. So, the total number of persons $= \ 6 + 15 + 6 - (1 + 1) - 2 = 23$.