Institute of Actuaries of India

Subject ST6 – Finance and Investment B

November 2008 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable

1

Year	Forward Rate	PV of $1 received at period end
1	5%	$1/1.05 = $0.9524
2	7%	$1/(1.05×1.07) = $0.8901
3	8%	$1/(1.05×1.07×1.08) = $0.8241

a. Price = ($60 × 0.9524) + ($60 × 0.8901) + ($1,060 × 0.8241) = $984.10

b. To show compute annuity-factor and pv-factor at 6.6% and show the present value is 984.10

$984.10 = [$60 × Annuity factor (6.6%, 3)] + [$1,000 × PV fa (6.65, 3)]

c. Next year, the price of the bond will be:

[$60 × Annuity factor (7%, 2)] + [$1,000 × PV factor (7%, $981.92

Therefore, there will be a capital loss equal to: $984.10 – $981. $2.18

The holding period return is: $\dfrac{\$60 + (-\$2.18)}{\$984.10} = 0.0588 = 5.88\%$

The answers differ because of the following reasons

i) holding period is less than the total and it is not a flat yield curve

ii) there is a change in the yield curve

They would have been equal if the IRR would have been equal to the flat yield curve and there is no change in the yield curve at the end of the period.

[5]

2 In each case, choose the longer-duration bond in order to benefit from a rate decrease.

a. The Aaa-rated bond will have the lower yield to maturity and therefore the longer duration.

b. The lower-coupon bond will have the longer duration *and* greater de facto protection from a call since coupon rates are also lowered at 4% compared to 8% coupon of the other bond

c. Choose the lower coupon bond for its longer duration.

[3]

3 a) Gamma believes that the market assessment of volatility is too high. Therefore, Gamma should sell options because the analysis suggests the options are overpriced with respect to true volatility. The delta of the call is 0.6, while that of the put is 0.6 – 1 = –0.4. Therefore, Gamma should sell puts and calls in the ratio of 0.6 to 0.4. For example, if Gamma sells 2 calls and 3 puts, the position will be delta neutral:

$$\text{Delta} = (2 \text{ x } 0.6) + [3 \times (-0.4)] = 0$$

b) The treasurer would like to buy the bonds today, but cannot. As a proxy for this purchase, T-bond futures contracts can be purchased. If rates do in fact fall, the treasurer will have to buy back the bonds for the sinking fund at prices higher than the prices at which they could be purchased today. However, the gains on the futures contracts will offset this higher cost to some extent. Also, this will not be accounted as a hedge under the regulation and will be treated as a naked derivative.

c) You would short Rs 0.50 of the market index contract and Rs 0.75 of the computer industry stock for each dollar held in Infosys.

[6]

4 (i) From put-call parity:

$$C = P + S_0 - Ke^{-rT} = 150 + 2500 - 2500e^{-0.06/6} = \text{Rs. } 174.88$$

(ii) Sell a straddle, i.e., sell a call and a put to realize premium income of:

Rs. 174.88 + Rs. 150 = Rs. 324.88

If the stock ends up at Rs. 2500, both of the options will be worthless and your profit will be Rs. 324.88. This is your maximum possible profit since, at any other stock price, you will have to pay off on either the call or the put. The stock price can move by Rs. 324.88 in either direction before your profits become negative.

(iii) Buy the call, sell (write) the put, lend: Rs. $2500e^{-0.06/6}$

The payoff is as follows:

Position	Immediate CF	CF in 2 months	
		$S_T \leq X$	$S_T > X$
Call (long)	C = 174.88	0	$S_T - 2500$
Put (short)	–P = 150	– (2500 – S	0
Lending position	$2500e^{-0.06/6}$	2500	2500

	$C - P + 2500e^{-0.06/6}$ = Rs.		
Total	2500	S_T	S_T

By the put-call parity theorem, the initial outlay equals the stock price:

S_0 = Rs. 2500

In either scenario, you end up with the same payoff as you would if you bought the stock itself.

[5]

5

Action	Cash Flows Now	T_1	T_2
Long futures with maturity T_1	0	$P_1 - F(T_1)$	0
Short futures with maturity T_2	0	0	$F(T_2) - P_2$
Buy asset at T_1, sell at T_2	0	$-P_1$	$+P_2$
At T_1, borrow $F(T_1)$	0	$F(T_1)$	$-F(T_1)e^{r_f(T_2-T_1)}$
Total	0	0	$F(T_2) - F(T_1)e^{r_f(T_2-T_1)}$

Where P_i is the price of the asset at time T_i for i= 1,2

b. Since the T_2 cash flow is riskless and the net investment was zero, then any non-zero profits would represent an arbitrage opportunity. If it is negative the reverse transactions would produce a positive profit.

c. The zero-profit no-arbitrage restriction implies that

$$F(T_2) = F(T_1)e^{r_f(T_2-T_1)}$$

[5]

6 a. The swap rate moved in favor of Alpha Limited. Alpha should have received 1% more per year than it could receive in the current swap market. Based on notional principal of Rs. 100 crores, the

loss is:

0.01 × Rs.100 crores = Rs. 1 crore per year.

b. The loss is 1 Cr every year for next 3 years and its market value can be computed using the swap rate and the three year discount rate. The value of the floating leg should equal the principal value and the value of the fixed leg should equal this value. Hence, if we can find the notional principal on which this 1 Cr is the interest amount then the present value of the fixed leg should be this principal notional amount and subtraction of the value of the principal should give us the value of the interest payments. Hence, the loss is:

1 Cr / .09 - 1 Cr / 0.09 * 1 / 1.1 ^3 = 2.7632 Cr

c. If Alpha had become insolvent, Beta would not be harmed. Beta would be happy to see the swap agreement cancelled. However, the swap agreement ought to be treated as an asset of Alpha when the firm is reorganized.

[6]

7 a) Risk-neutral world would mean that investors are indifferent to risks and hence they do not require any compensation for the risk and the expected return is the risk free rate. Under stochastic calculus this would imply that under risk-neutral measure the discounted processes are martingale.

b) This statement implies that the interest rate has a negative market price of risk. Since bond prices and interest rates are negatively correlated, the statement implies that the market price of risk for a bond price is positive. The statement is reasonable. When interest rates increase, there is a tendency for the stock market to decrease. This implies that interest rates have negative systematic risk, or equivalently that bond prices have positive systematic risk.

c) i) Note that in this world the discounted stock price process is martingale. Hence we can compute $d(B_t^{-1}e^{qt}S_t)$ where $B_t = \exp(rt)$ and S is as defined. Note that since S is giving a dividend at a rate q, we are assuming that we are reinvesting that dividend within S and hence at any point in time we are holding e^{qt} contracts of S. Using Ito we get,

$$d(B_t^{-1}e^{qt}S_t) = d(\exp(-(r-q)t)S_t)$$
$$= \exp(-(r-q)t)dS_t - (r-q)S_t\exp(-(r-q)t)dt$$
$$= \exp(-(r-q)t)S_t\left[(\mu-(r-q))dt + \sigma dz\right]$$

Since, this should be martingale implies that the drift term should be zero i.e, $\mu = (r-q)$ under the risk-neutral measure. Hence, the process followed by S under this world would be

$dS = (r-q)dt + \sigma dz$.

Market price of dz- risk is zero.

[8]

8 Suppose that S_1 is the stock price at time t_1 and S_T is the stock price at time T. Then under Black-Scholes world of risk-neutrality it implies that

$$\ln(S_1) - \ln(S_0) \approx N\left(\left(r_1 - \frac{\sigma_1^2}{2}\right)t_1,\ \sigma_1\sqrt{t_1}\right)$$

$$\ln(S_T) - \ln(S_1) \approx N\left(\left(r_2 - \frac{\sigma_2^2}{2}\right)t_2,\ \sigma_2\sqrt{t_2}\right)$$

where r's and σ's are constants.

Since the sum of the two independent normal distribution is normal with mean equal to the sum of the mean and variance equal to the sum of the variances it implies that

$$\ln(S_T) - \ln(S_0) \approx N\left(r_1t_1 + r_2t_2 - \frac{\sigma_1^2 t_1}{2} - \frac{\sigma_2^2 t_2}{2},\ \sqrt{\sigma_1^2 t_1 + \sigma_2^2 t_2}\right)$$

b) Because

$$r_1t_1 + r_2t_2 = \bar{r}\,T$$

and

$$\sigma_1^2 t_1 + \sigma_2^2 t_2 = \bar{V}\,T$$

it follows that

$$\ln(S_T) - \ln(S_0) \approx N\left(\left(\bar{r} - \frac{\bar{V}}{2}\right)T,\ \sqrt{\bar{V}\,T}\right)$$

c) If σ_i and r_i are the volatility and the risk-free interest rate during the i^th interval (i=1,2,3), an argument similar to that in a shows that:

$$\ln(S_T) - \ln(S_0) \approx N\left(r_1t_1 + r_2t_2 + r_3t_3 - \frac{\sigma_1^2 t_1}{2} - \frac{\sigma_2^2 t_2}{2} - \frac{\sigma_3^2 t_3}{2},\ \sqrt{\sigma_1^2 t_1 + \sigma_2^2 t_2 + \sigma_3^2 t_3}\right)$$

where t_1, t_2 and t_3 are the lengths of the three sub intervals. It follows that the result in b is still true.

d) The result in b remains true as the time between time zero and time T is divided into more subintervals each having its own risk free interest rate and volatility. In the limit it follows that if r and σ are known functions of time, the stock price distribution at time T is the same as that for a stock with a constant interest rate equal to the average interest rate and the constant variance rate equal to the average variance rate.

e) Under the Black-Scholes the option price is the expectation of a function of S_T under the risk-neutral measure. Since the distribution of S_T does not change (except for the parameters) the same black-scholes formula should be applicable with the replacement of the suitable parameters. Therefore, as long as the risk-free rate and volatility are known functions of time then the formula should still hold with appropriate parameter values.

[10]

9 a) Assume S_0, K, r, σ, T, q are the parameters for the over-the-counter option and S_0, K*, r, σ, T*, q are the parameters for the traded option. Suppose that d_1 has its usual meaning and is calculated on the basis of the first set of parameters while d_1* is the value of d_1 calculated on the basis of second set of parameters. Suppose further that w traded options are held for each over the counter option. The gamma of the portfolio is:

$$\alpha[\frac{\phi(d_1)e^{-qT}}{S_0\sigma\sqrt{T}} + w\frac{\phi(d_1*)e^{-qT*}}{S_0\sigma\sqrt{T*}}]$$

Where α is the number of over-the-counter options held.
Since we require gamma to be zero:

$$w = -\frac{\phi(d_1)e^{-q(T-T*)}}{\phi(d_1*)}\sqrt{\frac{T*}{T}}$$

The vega of the portfolio is:

$$\alpha\left[S_0\sqrt{T}\phi(d_1)e^{-qT} + wS_0\sqrt{T*}\phi(d_1*)e^{-qT*}\right]$$

Since we require vega to be zero

$$w = \sqrt{\frac{T}{T*}}\frac{\phi(d_1)e^{-q(T-T*)}}{\phi(d_1*)}$$

Equating the two expressions for w
T* = T
Hence the maturity of the option used for hedging must equal the maturity of the option being hedged.

b) When the asset price is positively correlated with volatility, the volatility tends to increase as the asset price increases, producing fat tails for high price and thin tails for low price. Implied volatility then increases with the strike price.

[7]

10 a) This is a binary call option.
$uS_0 = 110 \Rightarrow P_u = 1$

$dS_0 = 90 \Rightarrow P_d = 0$

The hedge ratio is: $H = \frac{P_u - P_d}{uS_0 - dS_0} = \frac{1-0}{110-90} = \frac{1}{20}$

A portfolio comprised of one share and short twenty binary options provide a guaranteed payoff of Rs.90, with present value: Rs. 90/1.05 = Rs. 85.71

Therefore:

S - 20P = Rs. 85.71

Rs. 100 - 20P = Rs. 85.71 $\Rightarrow$ P = Rs. 0.71

He would receive 0.71 mn if writes these options today.

b) Our goal is a portfolio with the same exposure to the stock as the hypothetical binary option. Since the option's hedge ratio is +0.05, the portfolio consists of 0.05 shares of stock, which costs Rs. 5, and by borrowing the present value of Rs 4.5 i.e. 4.29

Portfolio	S = 90	S = 110
Buy 0.05 shares	4.5	5.5
Borrow in T-bills	-4.50	-4.50
Total	0	1

This payoff is identical to that of the binary option. Thus, the stock plus borrowing strategy replicates both the cost and payoff of the binary option.

To lock in the 0.71 mn he should do the reverse, i.e. short 0.05mn shares, and Invest 4.29 mn in T-bills which would provide him the immediate cash of 0.71 mn with no bonus at maturity.

c) The binomial model is not a good model to estimate the value of the binary option since the binary option boundary may not coincide with one of the node of the binomial model. Also, particularly in this case, it is only a single step model – it should be large number of steps for it to provide a correct value.

[7]

11 The derivatives in question are over-the-counter (OTC) derivatives and are assets
available to back the liabilities to policyholders. The actuary would need to consider whether the value placed on these contracts adequately reflects their realisable values. Also, the mismatching risk between these assets and the liabilities has to be quantified and assessed. Consistency between the valuation of assets and liabilities is paramount in writing this type of business.

In the light of the death benefit, mortality risk has to be assessed and need to assess the suitability of any reinsurance contracts. The interaction of investment risk and mortality risk – even though small may be considered.

The actuary should also consider the extent to which the guarantee can be matched using the alternative assets available in the market and the implications of this for the value of the five OTC derivative contracts.

The actuary would need to consider the element of judgement involved in the models and input parameters used to price such derivatives. It is also important to consider the risk that the model proves inaccurate or does not capture some fundamental aspect of the market.

The actuary would need to consider whether the model is reasonable in the context of historical experience especially in relation to price movements in the Nifty. The actuary should be satisfied that the hedging techniques operated by the investment banks are adequate particularly if markets jump suddenly near the end of the life of the single premium bonds.

The actuary would need to have regard to the terms on which a part of the derivative contracts can be closed out prior to maturity to match lapses in the underlying liabilities and the frequency of such close outs.

The actuary needs to be aware that realisable value of the derivative contracts can change in value very quickly. He or she should not rely on valuations struck even a day ago in deciding surrender values. These should be based on the next available price matching the amount paid out to the realisable value of the part of the derivative closed out. This may involve surrenders waiting until the end of a calendar month or quarter before surrender can take place.

Checks should be carried out on the aggregation of counterparty risk and market exposure in order to ensure compliance with admissibility and other investment norms. This is particularly important in the context of counterparty exposure following large market movements. This is also closely related with credit risk of the derivative contracts and regular monitoring of this should be in place.

If there are interim settlements then need to ensure that robust processes are in place to carry that out.

The actuary would need to consider the ability of the company to meet its liabilities if there were a change in the tax treatment of the derivatives contracts in question.

Uncovered derivatives positions may arise from the failure to close out some part of the derivative contracts when payments have been made in respect of lapses. Such exposures must be carefully monitored as they are uncovered derivative positions. The impact of the derivative contracts on the overall position of the company must be considered by the appointed actuary.

The actuary may need to consider the adequacy of controls in the context of the use of derivatives. In particular, that the use is consistent with the objectives and policies laid down by the board for the use of derivatives.

He should give regard to the guidance notes issued by the institute and the regulations issued by the regulator while assessing the riskiness and the risk mitigation methods.

[10]

12 **Desirable features** are:

- reasonable dispersion of rates over time (due to the Brownian motion), otherwise there will be too large a probability of getting an absurdly high or low value market prices do not allow for these extremes [*another way of expressing this is to say that, when rates go too high or low, they tend to revert back to some middle level (mean-reverting)*]
- no negative interest rates
- easy to use and calculate, especially when calibrating to market prices
- bond (and/or swap) prices should be reproduced by the model
- forward rates should be imperfectly correlated although, this is only important when pricing certain types of option (e.g. yield spread options)
- volatility of rates of different maturity should be different, with generally shorter rates being the more volatile

The stochastic differential equation for the Vasicek model is

$$dr(t) = \alpha\left[\mu - r(t)\right]d(t) + \sigma d\tilde{W}(t)$$

First put all the terms containing r(t) on the left hand side:

$$dr(t) + \alpha r(t)dt = \alpha\mu d(t) + \sigma d\tilde{W}(t)$$

The above equation can be solved using integrating factor

$$\exp\left(\int \alpha dt\right) = e^{\alpha t}$$

Multiply both the side by the integrating factor we get

$$d\left[e^{\alpha t}r(t)\right] = \alpha\mu e^{\alpha t}d(t) + \sigma e^{\alpha t}d\tilde{W}(t)$$

Integrating the above over the range (t, T) we get,

$$\left[e^{\alpha u}r(u)\right]_t^T = \int_t^T \alpha\mu e^{\alpha u}d(u) + \int_t^T \sigma e^{\alpha u}d\tilde{W}(u)$$

i.e.

$$e^{\alpha T}r(T) - e^{\alpha t}r(t) = \mu\left[e^{\alpha T} - e^{\alpha t}\right] + \int_t^T \sigma e^{\alpha u}d\tilde{W}(u)$$

Rearranging and dividing through by $e^{\alpha T}$ gives:

$$r(T) = e^{-\alpha(T-t)} r(t) + \mu\left[1 - e^{-\alpha(T-t)}\right] + \int_t^T \sigma e^{-\alpha(T-t)} d\tilde{W}(u)$$

Under the risk neutral probability measure, $\tilde{W}(t)$ is standard brownian motion. So each increment $d\tilde{W}(u)$ is normally distributed with mean 0 and variance du. So, the Ito integral in the expression from part c is also normally distributed. It has mean zero. Its variance is

$$\operatorname{var}\left(\int_t^T \sigma e^{-\alpha(T-u)} d\tilde{W}(u)\right) = \int_t^T \sigma^2 e^{-2\alpha(T-u)} du = \left[\frac{\sigma^2}{2\alpha} e^{-2\alpha(T-u)}\right]_t^T = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(T-t)}\right)$$

So conditional distribution of r(T) given r(t) is normal with mean $e^{-\alpha(T-t)} r(t) + \mu\left[1 - e^{-\alpha(T-t)}\right]$ and variance $\frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(T-t)}\right)$.

[12]

13 Table 1: Unconditional default probabilities and survival probabilities

Time (Years)	Default Probability	Survival Probability
1	0.0300	0.9700
2	0.0291	0.9409
3	0.0282	0.9127
4	0.0274	0.8853
5	0.0266	0.8587

Table 2: Calculation of present value of expected payments (Payment = s per annum)

Time (Years)	Survival Probability	Expected Payment	Discount Factor	PV of Expected Payment
1	0.9700	0.9700s	0.9231	0.8954s
2	0.9409	0.9409s	0.8521	0.8018s
3	0.9127	0.9127s	0.7866	0.7179s
4	0.8853	0.8853s	0.7261	0.6429s
5	0.8587	0.8587s	0.6703	0.5756s
Total				3.6336s

Table 3: Calculation of the present value of expected payoff. Notional Principal =Re 1

Time (Years)	Default Probability	Recovery Rate	Expected Payoff	Discount Factor	PV of expected payoff
0.5	0.0300	0.3	0.0210	0.9608	0.0202
1.5	0.0291	0.3	0.0204	0.8869	0.0181
2.5	0.0282	0.3	0.0198	0.8187	0.0162

3.5	0.0274	0.3	0.0192	0.7558	0.0145
4.5	0.0266	0.3	0.0186	0.6977	0.0130
Total					0.0819

Table 4: Calculation of the present value of accrual payment

Time (Years)	Default Probability	Expected accrual payments	Discount Factor	PV of expected accrual payment
0.5	0.0300	0.0150s	0.9608	0.0144s
1.5	0.0291	0.0146s	0.8869	0.0129s
2.5	0.0282	0.0141s	0.8187	0.0116s
3.5	0.0274	0.0137s	0.7558	0.0103s
4.5	0.0266	0.0133s	0.6977	0.0093s
Total				0.0585s

From Table 2 and 4, the present value of expected payments is:
3.6336s + 0.0585s = 3.6921s
From Table 3, the present value of the expected payoff is 0.0819. Equatiing the two, we obtain the CDS spread
3.6921s = 0.0819
s = 0.0222
The mid-market spread should be 0.0222 times the principal or 222 basis points per year.

[10]

14 The function is integrable and measurable – the first term depends on the current values of X and Y with a simple polynomial function and the second term is an integral of Y over the past time period. Hence, the integrability and measurability is not an issue.
$E\left(X_t^2 Y_t \middle| F_s\right) = E\left(X_t^2 \middle| F_s\right) E\left(Y_t \middle| F_s\right) = \left(X_s^2 + t - s\right) Y_s$ (Using the result that $B_t^2 - t$ is a martingale – In fact this equation is analogous to this result – if we are allowed to substitute constant 1 in place of Y)

$$E\left(\int_0^t Y_u du \middle| F_s\right) = E\left(\int_0^s Y_u du \middle| F_s\right) + E\left(\int_s^t Y_u du \middle| F_s\right)$$

$$= \int_0^s Y_u du + Y_s (t-s)$$

Therefore,

$$E(Z_t | F_s) = \left(X_s^2 + t - s\right) Y_s - \left[\int_0^s Y_u du + Y_s (t-s)\right] = X_s^2 Y_s - \int_0^s Y_u du = Z_s$$

[6]
[100]

*******END*********