INSTITUTE OF ACTUARIES OF INDIA

CT1 – Financial Mathematics

OCTOBER 2009 EXAMINATION

INDICATIVE SOLUTION

General guidelines to markers:

The solutions provided here are indicative ones. Please award appropriate marks for any correct alternative solutions.

Please award marks for correct steps as indicated in the indicative solution even if the final answer does not match exactly.

If data input in a solution is wrong, please do not deduct more than 30% of maximum marks allocated to that part of the question.

Q.1 (i) Let P be the Equated Monthly Instalment.

Then $300{,}000 = 12\,P\,a^{(12)}_{\overline{12}|}$ @ 8% pa effective

$a_{\overline{12}|}$ @ 8% from Tables = 7.5361

$a^{(12)}_{\overline{12}|} = a_{\overline{12}|} * i/i(12) = 7.5361 * 1.036157 = 7.808583$

P = 3201.605

(ii) Loan o/s after 56 installments $= 12\,P\,a^{(12)}_{\overline{88/12}|}$ @ 8% pa

$= 12\,P\,a_{\overline{88/12}|} * i/i(12)$

= 12(3201.605)(5.391099)(1.036157)

= 214,610.95

Interest content of 57th installment $= 214{,}610.95 * i^{(12)}/12$

= 214,610.95 * 0.006434

= **1380.807**

Capital content of 57th installment = P – interest content

= 3201.605 – 1380.807

= **1820.798**

<u>Alternatively,</u>

Interest content of t th installment = P (1-v ^ (n-t+1)) @ $i^{(12)}/12$ (i.e.monthly rate)

Interest content of 57th installment = 3201.605*(1-v^(144-57+1)) @ 0.6434%

= 1380.809

Capital content of 57th installment = 3201.605*v^144-57+1) @ 0.6434%

= 1820.796

(iii) Loan o/s after 60th installment $= 12\,P\,a^{(12)}_{\overline{7}|}$ @ 8%

$= 12\,P\,a_{\overline{7}|} * i/i^{(12)}$ @ 8%

= 12(3201.605)(5.2064)(1.036157)

= **207,258.38**

(iv) Let initial installment = Q

$207{,}258.38 = Qv + Q(1.05)v^2 + Q(1.05)^2v^3 + Q(1.05)^3v^4 + Q(1.05)^4v^5$ @ 8%

$= Q/1.05\,\{1.05v + (1.05v)^2 + (1.05v)^3 + (1.05v)^4 + (1.05v)^5\}$ @ 8%

$= Q/1.05\; a_{\overline{5}|}$ @ i'

where 1/(1+i') = 1.05/1.08 ,

i' = 0.028571

= Q/1.05 * 4.59845

= **47,324.92**

(v) Let m be the number of monthly installments (i.e. in m/12 years)

$207{,}258.38 = 4000*12*a^{(12)}_{\overline{m/12}|}$ @ 8%

$= \dfrac{4000 * (1-v^{(m/12)})}{1.08^{(1/12)}-1}$

1,333.503 = 4000 * (1-v^(m/12))

1 – v^(m/12) = 0.333376

(m/12)log v = log(0.666624)

m = 63.23127

= 63 monthly installments of Rs.4000 each plus one residual payment in the 64th installment [total 64 installments]

(vi) Loan o/s after 63 installments $= 207{,}257.81(1.08)^{(63/12)} - 12*4000*s^{(12)}_{\overline{63/12}|}$ @ 8%

= 310,445.7 – 4000*12*6.448421

= 921.4542

Residual payment in the 64th installment = 921.4542*(1.006434)
[monthly rate of interest]
= 927.3828

Alternative method:-

Let z be the residual payment in the last instalment.

Therefore, the equation of value is

$4000\, a_{\overline{n}|} + z\, v^{n+1} = 207257.81$ @ 0.006434

$4000\, a_{\overline{63}|} + z\, v^{64} = 207257.81$ @ 0.006434

$a_{\overline{63}|} = 51.66$

$z\, v^{64} = 207257.81 - 206642.82 = 614.99$

=> $z = 614.99\,(1.006434)^{64} = \mathbf{927.10}$

[18]

[General comments on Q.1: This was reasonably well answered by all the candidates except part (iv) which only few could answer correctly.]

Q.2 (i) **Suppose a loan of Re.1 is taken at rate of interest i pa to be repaid after n years. Only interest payment is to be made during the n year period of i p.a.**

PV of interest payment $= i\, a_{\overline{n}|}$
PV of capital repaid $= v^n$
Loan = PV of interest + PV of capital

Hence, $1 = v^n + i\, a_{\overline{n}|}$

(ii) To Prove that $k\, \ddot{a}_{\overline{n}|} + (Ia)_{\overline{n-1}|} = (k-1)\ddot{a}_{\overline{n}|} + (I\ddot{a})_{\overline{n}|}$

LHS $= k\, \ddot{a}_{\overline{n}|} + (Ia)_{\overline{n-1}|}$

$= (k-1)\, \ddot{a}_{\overline{n}|} + \ddot{a}_{\overline{n}|} + (Ia)_{\overline{n-1}|}$ (By re-writing k as k-1+1)

$= (k-1)\, \ddot{a}_{\overline{n}|} + \dfrac{(1-v^n)}{d} + \dfrac{(\ddot{a}_{\overline{n-1}|} - (n-1)v^{n-1})}{i}$

$= (k-1)\, \ddot{a}_{\overline{n}|} + \dfrac{1-v^n+v(\ddot{a}_{\overline{n-1}|} - (n-1)v^{n-1})}{d}$ by using d=iv

$= (k-1)\, \ddot{a}_{\overline{n}|} + \dfrac{1-v^n+a_{\overline{n-1}|} - (n-1)v^{n}}{d}$ since $v\ddot{a}_{\overline{n-1}|} = a_{\overline{n-1}|}$

$= (k-1)\, \ddot{a}_{\overline{n}|} + \dfrac{1 - v^n + (v+v^2+v^3+\ldots\ldots+v^{n-1}) - (n-1)v^n}{d}$ by expanding $a_{\overline{n-1}|}$

$= (k-1)\, \ddot{a}_{\overline{n}|} + \dfrac{(1+v+v^2+v^3+\ldots\ldots+v^{n-1}) - v^n - (n-1)v^n}{d}$

$= (k-1)\, \ddot{a}_{\overline{n}|} + \dfrac{(\ddot{a}_{\overline{n}|} - n\,v^n)}{d}$

$= (k-1)\ddot{a}_{\overline{n}|} + (I\ddot{a})_{\overline{n}|}$

= RHS. Hence proved.

(iii) PV of (a) $= X\, v^{12}$
PV of (b) $= 1250\, v^n + 2500\, v^{2n} + 3750\, v^{3n}$
PV of (c) $= 6500\, v^{10}$
Given $v^n = 0.517541$
PV of (b) $= 1250(0.517541) + 2500*(0.517541)^2 + 3750(0.517541)^3$
$= 1836.38057$

Since PV of all three are equal, we get

$6500 v^{10} = 1836.38057$
$v^{10} = 1836.38057/6500 = 0.282520088$
$v = (0.282520088)^{\wedge}(1/10) = 0.881261776$
$1+i=1.13473661$
<u>i=13.47%</u>

PV of (c) = $X v^{12} = 1836.38087$
⇨ X = 1836.38087* (1.13473661)^12 = 8369.5766
<u>X = 8369.58</u> **[OR X = $6500*(1+i)^2$]**

(iv) Purchase price of annuity = PV of annuity + PV of expenses
PV of annuity = $100 [v+1.05v^2+1.05^2v^3+... + 1.05^9v^{10}]$ @ 9.2%
$= 100v_{9.2\%} [1+1.05v+1.05^2v^2+1.05^3v^3+.... +1.05^9v^9]$
$= 100/1.092 * \ddot{a}_{\overline{10|}}$ @ i'
where 1/(1+i') = 1.05/1.092
Hence i' = 4%
PV of annuity = $100/1.092 * (1.04)* a_{\overline{10|}}$ @ 4%
= 100/1.092 *1.04 *8.1109
= 772.4667

PV of expenses = $0.04*100[v+1.04(1.05)v^2+1.04^2 (1.05)^2 v^3+.............. + 1.04^9 (1.05)^9 v^{10}]$ @ 9.2%

$= \frac{4}{(1.04)(1.05)}[(1.04)(1.05)v + (1.04)^2(1.05)^2v^2 +....+ (1.04)^{10}(1.05)^{10}v^{10}]$

$= \frac{4}{(1.092)}[(1.092)v + (1.092)^2v^2 +....+ (1.092)^{10}v^{10}]$
4/1.092 (1+1+1 -------------10times)
= 4/1.092*10
= 36.6300

Purchase price of annuity =772.4667 + 36.63 = 809.0967
= Rs. 809.10

[16]

[General comments on Q.2: Many candidates could not give general reasoning in part (i). The algebraic proof in part (ii) was unnecessarily complicated by many. Part (iii) was well answered. Under part (iv) most of the candidates subtracted the present value of expenses from present value of annuity instead of adding it.]

Q.3 (i) Limitations of Immunization theory:

- **It may be necessary to rebalance the portfolio once interest rates have changed**
- **There may be options or other uncertainties in the assets or in the liabilities, making the assessment of the cashflows approximate rather than known**
- **Assets may not exist to provide the necessary overall asset volatility to match the liability volatility**
- **The theory relies upon small changes in interest rates. The fund may not be protected against large changes**
- **The theory assumes a flat yield curve and requires the same change in interest rates at all terms. In practice, this is rarely the case**
- **Immunization removes the likelihood of making large profits.**

(ii) Convexity of a series of cashflows of C_{tk} at time t_k where k=0,1,2...n is given as : n

$$\text{Convexity} = \frac{\sum_{k=1} C_{tk}\, t_k (t_k + 1) v^{tk+2}}{\sum_{k=1}^{n} C_{tk}\, v^{tk}}$$

(iii) PV of liabilities = $\sum 1000(1+e^{-t/100})v^t$ @ 5% ;t=1,2,3.....40

$= 1000(v^1+v^2+..\ v^{40})+1000(ve^{-1/100}+v^2e^{-2/100}+.....+ v^{40}\ e^{-40/100})$

$= 1000\ a_{\overline{40}|} + 1000\ \frac{k(1-k^{40})}{1-k}$ where $k = ve^{-1/100}$ and $k<1$ (Sum of a G.P.)

(0r $1000\ a_{\overline{40}|}$ @ 5%+1000 $a_{\overline{40}|}$ @ 6.0553%)

= 1000 (17.159086 + 14.942097)

= **32,101.183**

(iv) DMT of liabilities = $\frac{\sum 1000*t*(1+ e^{-t/100})v^t}{\sum 1000(1+ e^{-t/100})v^t}$ @ 5% ; t=1,2,3.....40

Numerator = $1000\ (Ia)_{\overline{40}|} + 1000 \sum te^{-t/100}v^t$ @ 5%

$= 1000\ (\ddot{a}_{40} - 40v^{40})/i + 1000(X \text{ say})$

where :

$X = e^{-1/100}v + 2e^{-2/100}v^2 + 3\ e^{-3/100}v^3 + ...\ + 40\ e^{-40/100}v^{40}$ @ 5%

$e^{-1/100}v = 0.9429046 = w$ say

Then $X = w + 2w^2 + 3w^3 + 4w^4 + ...\ + 40w^{40}$

$= (Ia)_{\overline{40}|}$ @ 6.0553%

$= (\ddot{a}_{\overline{40}|} - 40\ v^{40})/\ i$ @ 6.0553%

= (15.84688 – 3.808643)/0.060553

= 198.80604

∴Numerator = 1000 (18.01704 - 5.6818273)/.05 + 1000 X

= 1000 (246.704268 + 198.80604) = 445,510.308

Denominator = PV of liabilities = 32,101.183

DMT of liabilities = 445,510.308 / 32,101.183 = 13.8783

DMT = 13.88 years

(v) Let an amount P be invested in 20 year bonds and an amount Q be invested in 45 year bonds.

Given : PV of Assets = PV of liabilities

P + Q = 32101.183 --------- (Equation 1)

DMT of Assets= $\frac{0.05\ P\ (Ia)_{\overline{20}|} + 20\ P\ v^{20} + 0.05\ Q\ (Ia)_{\overline{45}|} + 45\ Q\ v^{45}}{\text{PV of Assets}}$

$= \frac{0.05\ P(110.950624) + P(7.537790) + 0.05Q(273.088608) + Q(5.008343)}{P+Q}$

$= \frac{13.08532\ P + 18.662773\ Q}{32101.183}$ from (Equation 1)

Given : DMT of Assets = DMT of liabilities

∴ 13.08532 P +18.662773 Q= 32101.183 * 13.8783 = 445510.3

Substituting P = 32101.183-Q from (1) in this equation, we get,

13.08532 (32101.183 – Q)+18.662773 Q = 445510.3

∴ Q = (445,510.3 – 420,054.268)/5.5774528 = 4564.1

and P = 27,537.1

Rs.27,537.1 is invested in 20 yr bonds &
Rs.4,564.1 is invested in 45 year bonds.

[25]

[General comments on Q.3: Part (i) being a bookwork was well answered. (ii) & (v) were poorly answered. Performance in parts (iii) & (iv), which were direct application of formulae followed by algebra, were not upto the mark.]

Q.4 (i) Suppose that Rs.100/- is invested in the deposit.
PV of interest + bonus interest (net of tax)=$(4\, a_{\overline{16}|} + 0.25*4*16)(1-0.20)$
PV of deposit returned at end of 4 years = $100v^{16}$

∴ The equation of value for the deposit is as under :-
100 = PV of(intt+bonus intt, net of tax) + PV of (deposit returned)
$= (4\, a_{\overline{16}|} + 0.25*4*16\, v^{16}) (1-0.20) + 100v^{16}$ @ i%
Where i is the net effective yield per quarter realized by the depositor.

Approximate i =(4 + 0.25*4)*0.80 = 4%
At i=4%, RHS = $(4*11.6523+16\, v^{16})(0.80)+53.39082 = 97.51219 <100$
So i<4%. At i=3% RHS = 110.4888.

Interpolating we get i=3.808% p.q. => yield p.a. = 16.13%

∴ **Net effective yield = 16.13% p.a.**

(ii) Rs.10,000 will earn 700 at the end of every year for 6 years.
Rs.700 will however accumulate @ 6% for the rest of the period.

∴ Accum. Value of 10,000 = $10,000 + 700\, s_{\overline{6}|}$ @ 6%
= 10,000+700(6.975319)=14882.72298

Similarly 100 will earn 7 at the end of every year while 7 will accumulate at 6%.

∴ Accum. Value of 100 of first year = $100 + 7\, s_{\overline{5}|}$ @ 6%
= 100+7(5.637093)=139.4596

Similary accumulated value of subsequent deposits (2nd to 5th years) is
$=100 + 7\, s_{\overline{4}|} +100 + 7\, s_{\overline{3}|} +100 + 7\, s_{\overline{2}|} +100 + 7\, s_{\overline{1}|}$
= 400+7(4.374616+3.1836+2.06+1) = 474.3275

Total accumulated value = 14882.72298 + 139.4596 + 474.3275
= **15496.51**

(iii) The force of interest t years after the start of the year is given by :

$$\delta(t) = 0.10 - \frac{(0.10 - 0.07)t}{9/12} \qquad 0 \le t \le 3/4$$

$$\delta(t) = 0.07 - \frac{(0.07 - 0.05)(t - 9/12)}{3/12} \qquad 3/4 \le t \le 1$$

The accumulation factor over the whole year is :

$$A(0,1) = \exp\left(\int_0^1 \delta(t)\, dt \right)$$

$$= \exp\left(\int_0^{0.75} (0.10 - 0.04t)dt + \int_{0.75}^{1} (0.13-0.08t)dt \right)$$

$$= \exp\left([0.10t - 0.02t]_0^{0.75} + [0.13t - 0.04t]_{0.75}^{1} \right)$$

$$= \exp(0.075 - 0.01125 + 0.325 - 0.0175) = \exp(0.07875) = 1.0819$$

Accumulated value of 10,000 = 10,000 * 1.0819 = 10,819

[13]

[General comments on Q.4: This question was the most poorly answered of all. Candidates need to demonstrate higher order skills to solve these.]

Q.5 (i) **The discounted payback period is the smallest time t for which the accumulated value of the revenue up to time t exceeds the accumulated value of the costs up to time t.**

(ii) **The payback period is the same as discounted payback period, except that the accumulation is carried out using an interest rate of 0. In other words, it is the earliest time for which the monetary value of the revenue exceeds the monetary value of the costs.**

(iii) **Let t be the discounted payback period. Working in crores, equation of value is**

$$-500(1.1)^t - 1500(1.1)^{(t-0.5)} + 420(1.1)^{(t-2)} + 180\, \bar{s}_{\overline{1}|}(1.1)^{(t-3.5)} + 1200\, \bar{s}_{\overline{(t-3.5)}|} = 0$$

$$\bar{s}_{\overline{1}|} = 1.049206$$

$$\Rightarrow -500(1.1)^t - 1500(1.1)^{(t-0.5)} + 420(1.1)^{(t-2)} + 188.85708\ \bar{s}_{\overline{1}|}\ (1.1)^{(t-3.5)} + 12590.49\,[(1.1)^{(t-3.5)} - 1] = 0$$

For t = 6, LHS = 822.7175

For t = 5, LHS = -396.665

$$\text{Using interpolation, } t = 0.05 + \frac{-396.665 - 0}{-396.665 - 822.7175}(0.06 - 0.05)$$

$$\Rightarrow t = 5.34 \text{ years}$$

Alternative method:-

Using present value method,

$$-500 - 1500v^{0.5} + 420v^2 + 180\,\bar{a}_{\overline{1}|}v^{2.5} + 1200\,\bar{a}_{\overline{(t-3.5)}|}v^{3.5} = 0 \quad @10\%$$

$$-1930.19 + 482.395 + 1200\,\bar{a}_{\overline{(t-3.5)}|}v^{3.5} = 0$$

$$\bar{a}_{\overline{(t-3.5)}|} = \frac{1447.795}{1200}(1.1)^{3.5} = 1.684225$$

$$1 - v^{(t-3.5)} = 0.16052383$$

$$t - 3.5 = 1.83587$$

$$t = 5.336 \text{ years}$$

[10]

[General comments on Q.5: The definitions were well answered. But most students made mistake in DPP equation. Some of them did not work in crores and carried full digits throughout and made the working very clumsy.]

Q.6 **Solution :-**

(a) **Given $E(i_t) = .08$ and $V(i_t) = .07^2$**
i.e., j = .08 and s = .07 using usual notations.

Required to calculate $1000.E[S_{15}]$ and $1000. V[S_{15}]^{1/2}$
$E[S_{15}] = (1+j)^{15} = 1.08^{15} = 3.172169$
$1000.E[S_{15}] = 3172.17$
Expected value of accumulation of Rs.1000 after 15 years = 3172.17

$V[S_{15}] = (1+j^2 + 2j + s^2)^{15} - (1+j)^{30}$
$= (1+.08^2 + 2*.08 + .07^2)^{15} - (1.08)^{30} = 0.653083$
$S.D[S_{15}] = V[S_{15}]^{1/2} = 0.808135$

Standard Deviation of accumulation of Rs.1000 after 15 years = 808.135

(b) **Expected accumulation of a single investment of Re.1 = $E[S_{15}]$**
$=(1+j)^{15} = 1.08^{15} = 3.172169$

If $1+i_t \sim \log N(\mu, \sigma^2)$ then
$E[1+i_t] = \exp(\mu + \sigma^2/2)$ and $V[1+i_t] = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1]$
i.e., $E[i_t] = \exp(\mu+\sigma^2/2) - 1$ and $V[i_t] = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1]$

We know that $E(i_t) = .08$ and $V(i_t) = .07^2$
∴ We get the two equations :-

$\exp(\mu + \sigma^2/2) - 1 = .08$
$\exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1] = .07^2$

$\Rightarrow \quad 1.08^2 [\exp(\sigma^2) - 1] = .07^2$
$\exp(\sigma^2) = 1.004201 \Rightarrow \sigma^2 = .004192 \Rightarrow \underline{\sigma = 0.064747}$
and, $\underline{\mu = 0.074865}$

(c) **Using multiplicative of lognormal random variables,**
$S_{15} \sim \log N(15\mu, 15\sigma^2)$ or $(\log S_{15} - 15\mu)/\sqrt{15}\,\sigma \sim N(0,1)$

Prob that accumulation of Re.1 will
be less than 60% of Exp value(3.172169)= $P(S_{15} \le 1.903301)$
$= P((\log S_{15} - 15\mu)/\sqrt{15}\,\sigma \le (\ln(1.903301) - 15\mu)/\sqrt{15}\,\sigma)$
$= P(Z \le -1.9117) = 0.028$

[10]

[General comments on Q.6: While the parameters were calculated correctly by majority of candidates, parts (i) & (iii) were not well answered.]

Q.7 **(i)** **A forward contract is an agreement made between two parties under which one agrees to buy from the other a specified amount of an asset at a specified price, called the forward price, on a specified future date.**

The investor agreeing to sell the asset is said to hold a short forward position in the asset, and the buyer is said to hold a long forward position.

(ii) **a) The forward price for a security with no income is:**
$K = S_o e^{\delta T}$
where S_o is the price of the underlying asset at time 0,
δ is the risk free force of interest
T is the time the contract matures i.e. when the sale actually happens.

b) The forward price for a security with fixed cash income is:
$K = (S_o - I) e^{\delta T}$

where I is the present value of income received during the term of the contract.

c) The forward price for a security with a known dividend yield, D, where dividends are payable continuously is:

$$K = S_o e^{(\delta - D)T}$$

(iii) The term structure of interest rates refers to the variation by term of interest rates.

(iv) The theories that explain the term structure of interest rates are:

1) **Expectations Theory: It states that the relative attraction of short and longer-term investments will vary according to expectations of future movements in interest rates. An expectation of a fall in interest rates will make short term investments less attractive and longer-term investments more attractive. An expectation of a rise in interest rates will have the converse effect.**
2) **Liquidity Preference Theory: It states that, as a general rule, investors prefer to hold stocks that they are not "locked into" for a long period.**
   They are therefore prepared to pay more for shorter, more liquid, stocks.
   This implies that the yield obtainable on these shorter stocks will be lower.
3) **Market Segmentation Theory: It states that different forces of supply and demand affect the term structure of interest rates. Bonds of different terms are attractive to different investors, who will choose assets that are similar in term to their liabilities.**

[8]

[General comments on Q.7: This being a direct question from study material, was well answered.]

Total 100 Marks